# Tutorial 11: Selected problems of Assignment 10

Leon Li

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# Arzelà-Ascoli Theorem

**Thm** $\mathcal{E} \subseteq (C[a,b], \|\cdot\|_\infty)$ is precompact $\Leftrightarrow$ $\mathcal{E}$ is bounded and equicontinuous
(precpt) (bdd) (equicts)

Application to sequences: given $(f_n) \subseteq C[a,b]$, $\mathcal{E} := \{f_n \mid n \in \mathbb{N}\} \subseteq C[a,b]$
(as a seq.) (as a set)

**Cor 1** If $\mathcal{E}$ is bdd and equicts, then $(f_n)$ has a convergent subsequence
(conv. subseq.)

An application of Corollary 1: Compact operator

**Def** Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces, a linear map $T: X \to Y$

is compact (cpt) if $\forall B \subseteq X$: bdd subset, $T(B) \subseteq Y$ is precpt.

Equivalently: $\forall (x_n) \subseteq X$: bdd seq, $(Tx_n) \subseteq Y$ has a conv. subseq.

**Cor 2** $(X, \|\cdot\|_X) = (C[a,b], \|\cdot\|)$ w/ arbitrary norm $\|\cdot\|$, $(Y, \|\cdot\|_Y) = (C[a,b], \|\cdot\|_\infty)$

If $\forall (f_n) \subseteq (C[a,b], \|\cdot\|)$: bdd, $\{Tf_n\} \subseteq (C[a,b], \|\cdot\|_\infty)$ is bdd and equicts

then $T: C[a,b] \to C[a,b]$ is cpt.

**Pf:** The result follows from Cor 1.

Q1) (HW10, Q6) Define $T : C[0,1] \to C[0,1]$ by $T(f)(x) := \int_0^x f(t)\,dt$

(a) Show that $T : (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$ is cpt

(b) Show that $T : (C[0,1], \|\cdot\|_2) \to (C[0,1], \|\cdot\|_\infty)$ is cpt

Sol) (a) Given $(f_n) \subseteq (C[0,1], \|\cdot\|_\infty)$ bdd, $\exists M > 0$ s.t.

$$\|f_n\|_\infty \le M, \quad \forall n \in \mathbb{N}$$

(i) $\{Tf_n\} \subseteq (C[0,1], \|\cdot\|_\infty)$ is bdd : $\forall n \in \mathbb{N}$, $\forall x \in [0,1]$, $|(Tf_n)(x)|$

$= \left|\int_0^x f_n(t)\,dt\right| \le \|f_n\|_\infty (x-0) \le M$, $\quad \therefore \|Tf_n\|_\infty \le M$, $\forall n \in \mathbb{N}$.

(ii) $\{Tf_n\}$ is equicts : showing $\{Tf_n\}$ satisfies a uniform Lipschitz condition :

$\forall x, y \in [0,1]$ (WLOG $x \ge y$), $\forall n \in \mathbb{N}$, $|Tf_n(x) - Tf_n(y)| = \left|\int_y^x f_n(t)\,dt\right|$

$\le \|f_n\|_\infty |x-y| \le M|x-y|$

$\therefore$ By Cor 2, $T$ is cpt.

(b) Given $(f_n) \subseteq (C[0,1], \|\cdot\|_2)$ bdd, $\exists K > 0$ s.t.

$$\|f_n\|_2^2 = \int_0^1 |f_n(t)|^2 dt \le K, \ \forall n \in \mathbb{N}$$

(i) $\{Tf_n\} \subseteq (C[0,1], \|\cdot\|_\infty)$ is bdd : $\forall n \in \mathbb{N}$, $\forall x \in [0,1]$,

$$|Tf_n(x)| = \left| \int_0^x f_n(t)\, dt \right| \le \int_0^1 |f_n(t)||1|\, dt \underset{\text{(Cauchy-Schwarz ineq.)}}{\le} \left( \int_0^1 |f_n(t)|^2 \right)^{\frac{1}{2}} \left( \int_0^1 |1|^2 dt \right)^{\frac{1}{2}} \le \sqrt{K}$$

$$\therefore \|Tf_n\|_\infty \le \sqrt{K}, \ \forall n \in \mathbb{N}.$$

(ii) $\{Tf_n\}$ is equicts : showing $\{Tf_n\}$ satisfies a uniform Hölder condition :

$\forall x, y \in [0,1]$ (WLOG $x \ge y$), $\forall n \in \mathbb{N}$, $|Tf_n(x) - Tf_n(y)| = \left| \int_y^x f_n(t) dt \right|$

$$\le \int_y^x |f_n(t)||1| dt \underset{\text{(Cauchy-Schwarz ineq.)}}{\le} \left( \int_y^x |f_n(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_y^x |1|^2 dt \right)^{\frac{1}{2}} \le \sqrt{K}\, |x-y|^{\frac{1}{2}}$$

$\therefore$ By Cor 2, $T$ is cpt. $\quad -\boxed{\mathbb{Z}}$

Q2) (HW10, Q8) Fix $K(x,y) \in C([a,b]\times[a,b])$, $\forall f \in C[a,b]$, define

the integral transform $Tf : [a,b] \to \mathbb{R}$ by $(Tf)(x) = \int_a^b K(x,y) f(y)\,dy$

(a) Show that $T$ defines a linear operator $T : C[a,b] \to C[a,b]$.

(b) Show that $T : (C[a,b], \|\cdot\|_\infty) \to (C[a,b], \|\cdot\|_\infty)$ is cpt.

Sol) (a) Showing $Tf \in C[a,b]$: $\forall \varepsilon > 0$, by uniform cty of $K$.

$\exists \delta > 0$ s.t. $\forall x, x' \in [a,b]$ w/ $|x - x'| < \delta$, $\forall y \in [a,b]$, $|K(x,y) - K(x',y)| < \varepsilon$

$\therefore$ $|Tf(x) - Tf(x')| = \left| \int_a^b (K(x,y) - K(x',y)) f(y)\,dy \right| \leq \varepsilon \cdot \|f\|_\infty (b-a)$

Linearity of $T$ follows from the linearity of integration.

(b) Given $(f_n) \subseteq (C[a,b], \|\cdot\|_\infty)$ bdd, $\exists M > 0$ s.t. $\|f_n\|_\infty \leq M$, $\forall n \in \mathbb{N}$

(i) $\{Tf_n\} \subseteq (C[a,b], \|\cdot\|_\infty)$ is bdd: $\forall n \in \mathbb{N}$, $\forall x \in [a,b]$, $|(Tf_n)(x)|$

$= \left| \int_a^b K(x,y) f_n(y)\,dy \right| \leq \|K\|_\infty \|f_n\|_\infty (b-a) \leq \|K\|_\infty M (b-a)$

(ii) $\{Tf_n\}$ is equicts: Under same notations as in (a), $\exists \delta > 0$, $\forall n \in \mathbb{N}$,

$$|Tf_n(x) - Tf_n(x')| \leq \varepsilon \cdot \|f_n\|_\infty (b-a) \leq \varepsilon M (b-a)$$

$\therefore$ By Cor 2, $T$ is cpt. $-\boxtimes$

Q3) (HW10, Q4) Let $\mathcal{E} = \{f_1, \cdots, f_N\} \subseteq C[a,b]$ be a finite set, show that $\mathcal{E}$ is bdd and equicts.

Sol 1) By definition: (1) $\mathcal{E}$ is bdd:

Let $M := \max_{1\le i\le N}\{\|f_i\|_\infty\}$, then $\forall\, 1\le i\in N$, $\|f_i\|_\infty \le M$

(2) $\mathcal{E}$ is equicts: $\forall \varepsilon>0$, since $\forall\, 1\le i\le N$, $f_i$ is uniformly cts,

$\exists\, \delta_i > 0$ s.t. $\forall x,y\in[a,b]$ w/ $|x-y|<\delta_i$, $|f_i(x)-f_i(y)|<\varepsilon$

Let $\delta := \min_{1\le i\le N} \delta_i$, then $\forall x,y\in[a,b]$ w/ $|x-y|<\delta$, $\forall\, 1\le i\le N$,

$$|f_i(x)-f_i(y)|<\varepsilon$$

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Sol 2) By Thm, suffices to show $\mathcal{E}$ is precpt:

$\forall (g_n)\subseteq \mathcal{E}$, since $\mathcal{E}$ is finite, $(g_n)$ contain a subsequence $(g_{n_k})$ of the form $g_{n_k} = f_i$, $\exists\, 1\le i\le N$, $\forall k\in\mathbb{N}$

$\therefore (g_{n_k})$ converges uniformly to $f_i$, hence is convergent in $(C[a,b], \|\cdot\|_\infty)$

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